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XIII.

VARIABLE STARS OF SHORT PERIOD.

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IN a recent communication to this Academy * the following classification of the variable stars was proposed: —

I. Temporary stars. Examples, Tycho Brahé's star of 1572, new star in Corona, 1866.

II. Stars undergoing great variations in light in periods of several months or years. Examples, *o Ceti* and *χ Cygni*.

III. Stars undergoing slight changes according to laws as yet unknown. Examples, *α Orionis* and *α Cassiopeiæ*.

IV. Stars whose light is continually varying, but the changes are repeated with great regularity in a period not exceeding a few days. Examples, *β Lyræ* and *δ Cephei*.

V. Stars which every few days undergo for a few hours a remarkable diminution in light, this phenomenon recurring with great regularity. Examples, *β Persei* and *S Cancri*.

A discussion was given, in the article referred to, of the stars of the last class. It was shown that in the case of *β Persei* at least, the observed variations could be very satisfactorily explained by the theory that the reduction in light was caused by a dark eclipsing satellite. The dimensions of this satellite and of its orbit were then computed. The variations of the stars of the fourth class will be considered in the present paper. Both of these papers must be regarded as preliminary, rather than final, discussions. Observations are now in progress at the Harvard College Observatory which greatly increase the precision of our knowledge of many of the constants involved. When these are completed, a revision of the whole investigation is much to be desired. To avoid all prejudice, the present papers are made to

* Proc. Amer. Acad., xvi. 1.

depend entirely on the work of previous observers. Approximate methods are depended upon throughout, where a rigorous computation would have been employed, if the results were to be regarded as final. In Table I. is given a list of all the known variable stars whose periods are less than three months. The successive columns give a current number, the number in Schönfeld's Second Catalogue,* the name of the star, and the class to which it belongs, when this is known with certainty. Then follow the right ascension and declination for 1880, the period in days, and the magnitudes at maximum and minimum. The data for the southern stars which are not given by Schönfeld are taken from the Uranometria Argentina. The last columns give the name of the discoverer and the year in which the variability was detected.

TABLE I. — Variable Stars of Short Periods.

Number.	Schön.	Name.	Cl.	R. A. 1880.	Dec. 1880.	Period.	Max.	Min.	Discoverer.	Year.
				h. m.	° ′					
1	..	— Cephei	V	0 51.7	+81 14	2.49	7	10	Ceraski	1880
2	17	β Persei	V	3 0.4	+40 30	2.87	2.2	3.7	Montanari	1669
3	19	λ Tauri	V	3 54.0	+12 9	3.95	3.4	4.2	Baxendell	1848
4	32	*T* Monocerotis	IV	6 18.7	+ 7 9	27.00	6.2	7.6	Gould	1871
5	34	*S* Monocerotis	..	6 34.4	+10 0	3.40	4.9	5.4	Winnecke	1867
6	36	ζ Geminorum	IV	6 57.0	+20 45	10.16	3.7	4.5	Schmidt	1844
7	..	*U* Monocerotis	..	7 25.1	— 9 32	46.00	6.0	7.2	Gould	1873
8	47	*S* Cancri	V	8 37.1	+19 28	9.48	8.2	9.8	Hind	1848
9	..	*N* Velorum	..	9 27.6	—56 30	4.25	3.4	4.4	Gould	1871
10	..	*l* Carinæ	..	9 42.0	—61 57	31.25	3.7	5.2	Gould	1871
11	..	*R* Muscæ	..	12 34.8	—68 45	0.89	6.6	7.4	Gould	1871
12	66	*W* Virginis	..	13 19.8	— 2 45	17.27	8.7–9.2	9.8–10.4	Schönfeld	1866
13	74	δ Libræ	V	14 54.6	— 8 2	2.32	4.9	6.1	Schmidt	1859
14	..	*T* Triang. Aust.	..	14 58.6	—68 15	1.00	7.0	7.4	Gould	1871
15	..	*R* Triang. Aust.	..	15 9.1	—66 3	3.40	6.6	7.5	Gould	1871
16	75	*U* Coronæ	V	15 13.3	+32 5	3.45	7.6	8.8	Winnecke	1869
17	96	*u* Herculis	..	17 12.9	+33 14	38.50	4.6	5.4	Schmidt	1869
18	98	*X* Sagittarii	..	17 40.0	—27 47	7.01	4	6	Schmidt	1866
19	99	*W* Sagittarii	..	17 57.4	—29 35	7.59	5	6.5	Schmidt	1866
20	..	— Sagittarii	..	18 9.8	—34 9	2.42	6.2	7.4	Gould	1871
21	103	*U* Sagittarii	..	18 24.8	—19 13	6.75	7.0	8.3	Schmidt	1866
22	105	*R* Scuti	..	18 41.1	— 5 50	71.10	4.7–5.7	6.0–8.5	Pigott	1795
23	..	κ Pavonis	..	18 44.6	—67 23	9.10	4.0	5.5	Gould	1872
24	106	β Lyræ	IV	18 45.6	+33 13	12.91	3.4	4.5	Goodricke	1784
25	107	*R* Lyræ	..	18 51.7	+43 47	46.00	4.3	4.6	Baxendell	1856
26	108	*S* Coron. Aust.	..	18 53.1	—37 7	6.20	9.8	11.5?	Schmidt	1866
27	109	*R* Coron. Aust.	..	18 53.8	—37 7	54.00	10.5–11.5	<12.5	Schmidt	1866
28	116	*S* Vulpeculæ	..	19 43.5	+26 59	67.50	8.4–8.9	9.0–9.5	Rogerson	1837
29	118	η Aquilæ	IV	19 46.4	+ 0 42	7.18	3.5	4.7	Pigott	1784
30	122	*R* Sagittæ	..	20 8.6	+15 22	70.42	8.5–8.7	9.8–10.4	Baxendell	1859
31	137	δ Cephei	IV	22 24.7	+57 48	5.37	3.7	4.9	Goodricke	1784

* Zweiter Catalog von veränderlichen Sternen. Mannheim, 1875.

From this table it appears that only six stars of the fifth class are as yet known. Although the published observations of some of the others are insufficient to determine the nature of their variations, it is probable that most of them belong to the fourth class. The first star on the list, which is DM. 81°.25, has been designated as *T Cephei*,* but the use of this name has created much confusion. In 1863,† Argelander announced the variability of the star DM. 55°.2943, and this star is called *T Cephei* in Chambers' Astronomy, p. 586. In 1879, Ceraski ‡ announced that DM. 67°.1291 was variable. When correcting its position § he called it *T Cephei*. This correction is quoted in the "Astronomical Register," xviii. 322, under the heading "W. Ceraski's new Variable," apparently confounding it with Ceraski's last discovery, DM. 81°.25.

The most natural explanation of the variation of a star of short period is that it is due to its rotation around its axis.

In the Annals of the Harvard College Observatory, xi. 264, the variation in light of Iapetus, the outer satellite of Saturn, is discussed on this hypothesis. It is there shown that if the axis of revolution is perpendicular to the line of sight, the variation of light, L, may be approximately represented by the formula, $L = a + b \sin v + c \cos v + d \sin 2v + e \cos 2v$; a here denotes the mean light, v the angle of rotation, b and c are constants depending on the comparative brilliancy of the two hemispheres, each of which is supposed to be of uniform intensity, but one brighter than the other; d and e depend on a supposed deviation of the body from the form of a solid of revolution. This equation may also be written in the form $L = a + m \sin (v + \alpha) + n \sin (2v + \beta)$, in which α depends upon the angular position of the plane separating the two hemispheres from the line of sight at the epoch from which the variation in light is reckoned; β in like manner depends upon the positions in which the body subtends its largest and smallest discs. Our problem then is to see how far this equation will represent the variation in light of all the stars of the fourth class.

Both of these proposed causes of variation may be criticised as improbable; but what could be more improbable than the phenomenon itself, were it not verified by observation? With our present knowl-

* Science Observer, iii. 30, 38, 48. English Mechanic and World of Science, xxxii. 297. Astron. Nach. xcix. 87.

† Astron. Nach., lxi. 281.

‡ Astron. Nach., xciv. 175.

§ Astron. Nach., xcviii. 239.

edge of the constitution of the stars it would seem extremely unlikely that certain of them would lose half their light at regular intervals of from one to twelve days. If it can be shown that the hypothesis satisfies the observed facts, it seems unreasonable to deny it until some more probable explanation can be offered. The difference in brightness of the two sides of a star may be due to spots like those of our sun, to large dark patches, or to a difference in temperature. In the latter case, observations of the distribution in light in the spectrum at the maxima and minima might show a greater variation in the blue than in the red portions. If the body had the form of an oblate ellipsoid rotating around one of its longer axes, its condition of equilibrium would be unstable. If, however, it was a prolate ellipsoid it would be in stable equilibrium, and if sufficiently rigid might revolve in this way indefinitely. If, like our sun, it was in a fluid condition, we might anticipate a return to the form of a solid of revolution. Jacobi has however shown * that a fluid ellipsoid having three unequal axes may be in equilibrium when revolving around its shortest axis. An analogous case is found in Plateau's experiment, where a globule of oil suspended in alcohol and water is made to revolve. With a sufficient velocity the globule, if slightly eccentric, elongates before throwing off a satellite. We may also assume the existence of two nuclei, or that the two components of a binary star are so close together that both are enveloped in the incandescent gas or photosphere.

Another equation of condition would thus be furnished which might serve to determine the absolute diameter of the star in miles. Thus the observations discussed below give the relative dimensions of two of the axes, and the condition that the body shall be in equilibrium will determine the relative length of the axis of revolution. If the star was an ellipsoid of revolution we could compute the flattening at the poles from the diameter and the time of revolution; we could also compute the diameter if the other two constants were given. Although the problem is more complex, evidently the same principle may be applied to an ellipsoid with three unequal axes.

Four of these stars, ζ *Geminorum*, β *Lyræ*, η *Aquilæ*, and δ *Cephei*, have been observed with great care, so that their variations are known with much precision. Each will therefore be discussed in turn, according to the following method. As the variation is periodic, it will be convenient to denote the time by an angle, v, such that 360° shall correspond to one period or revolution of the star. We now wish the light

* Poggendorff's Annalen, xxxiii 229; see also Journ. Frank. Inst. cx. 217.

corresponding to $v = 0°, 15°, 30°, 45°$, etc. The period is divided into twenty-four equal parts, and the number of grades corresponding to each is taken from the light curves. Argelander and Schönfeld give the number of grades for each hour, and the number of grades was found from their tables by interpolation. Oudemans represents his results graphically, and the grades were taken from his curves by inspection. These curves were used rather than the original observations, in order to reduce the accidental errors. The results are free from prejudice, since they were drawn by the observers themselves without regard to any theory. They are, however, open to the objection that small systematic errors may be present which are not easily detected.

We must next pass from grades to the actual intensities of light. For this purpose we cannot rely on an assumed value of a grade, since we have no certainty that this will be the same for lights of different intensity. Accordingly, the comparison stars used by each observer are compared with the measures of Wolff.* Points were constructed whose abscissæ equal the assumed light of each comparison star in grades, and their ordinates, the logarithms of the light as measured by Wolff. A smooth curve was then drawn through these points, and served to convert the grades into logarithms. The readings are only made to hundredths, since one unit in this place corresponds to one-fortieth of a magnitude, and all the curves are uncertain by much more than this amount. The largest logarithm is then subtracted from all the others, and the number corresponding to the difference gives the intensity of the light. This is multiplied by one hundred, so that the results are given in percentages.

In the equation, $L = a + m \sin (v + \alpha) + n \sin (2v + \beta)$, a is found from the mean of all the observed values of L. The other constants might be found from a solution by least squares, forming twenty-four equations of condition with the twenty-four deduced values of L. Sufficient accuracy is, however, obtained by approximate graphical methods. By adding together the values of L, corresponding to values of v differing 180°, we eliminate the term $m \sin (v + \alpha)$. Their differences, in like manner, eliminate $n \sin (2v + \beta)$. Each then may thus be found independently of the others.

ζ Geminorum. In 1848, Argelander gave a light curve of this star.†

* Photometrische Beobachtungen an Fixsternen, Leipzig, 1877.
† Astron. Nach., xxviii. 33.

Table II. gives the data for converting the grades into light ratios by means of the comparison stars. The successive columns give the name of the star, its light in logarithms as measured by Wolff, its light in grades assumed by Argelander, the corresponding ordinate of the curve, and the second column minus the fourth, or the assumed errors.

TABLE II. — COMPARISON STARS FOR ζ GEMINORUM.

Name.	Wolff.	Grades.	Curve.	$W - C$
ξ Geminorum	8.81	9.9	8.81	.00
δ Geminorum	8.73	8.8	8.74	—.01
λ Geminorum	8.68	8.0	8.70	—.02
ι Geminorum	8.66	6.0	8.64	+.02
υ Geminorum	8.58	3.3	8.58	.00
ν Geminorum	8.55	2.0	8.55	.00

The curve here agrees very well with the measurements, but its inclination is much greater for the brighter than for the fainter stars. In other words, a grade represents a much larger difference in magnitude when the star is bright than when faint. The change is, however, slight between the limits within which the curve is used.

Table III. gives a comparison of the light curve with theory. The successive columns give the angle v, the corresponding time from the minimum, and the observed light in grades, in logarithms, and in percentages. The next column gives the light computed by the formula, $L = 89.6 + 10.0 \sin v$; this is followed by the residuals found by subtracting the computed from the observed brightness. As they show a perceptible systematic error, two more columns are given corresponding to the formula $L = 89.6 + 10.2 \sin (v - 11^\circ.3)$. This gives an entirely satisfactory agreement with observation, the average deviation amounting to less than one per cent. It cannot be reduced directly to magnitudes, since, when the light equals 100, one per cent equals .011 magnitudes, when 80, .014 magnitudes, and when 50, .022 magnitudes. The average deviation is accordingly only about one hundredth of a magnitude. Since two smooth curves are compared, the small irregular variations in the residuals are principally due to the neglected thousandths in the logarithm of the light. They are, however, probably far less than the real errors of the curves. The mean of the residuals is given in the last line of the table.

TABLE III. — VARIATION IN LIGHT OF ζ GEMINORUM.

v.	Time.	Gr.	Log.	Obs.	Comp.	$O - C$	Comp.	$O - C$
°	d. h.							
0	0 0.0	3.4	8.58	79	80	—1	79	0
15	0 10.2	3.6	8.58	79	80	—1	80	—1
30	0 20.3	4.1	8.59	81	81	0	82	—1
45	1 6.5	4.7	8.61	85	82	+3	84	+1
60	1 16.6	5.2	8.62	87	85	+2	86	+1
75	2 2.8	5.7	8.63	89	87	+2	89	0
90	2 12.9	6.1	8.64	91	90	+1	92	—1
105	2 23.0	6.5	8.65	93	92	+1	94	—1
120	3 9.2	6.9	8.66	96	95	+1	96	0
135	3 19.4	7.2	8.67	98	97	+1	98	0
150	4 5.5	7.3	8.67	98	98	0	100	—2
165	4 15.7	7.4	8.68	100	99	+1	100	0
180	5 1.8	7.4	8.68	100	100	0	100	0
195	5 12.0	7.3	8.67	98	99	—1	99	—1
210	5 22.2	7.1	8.67	98	98	0	98	0
225	6 8.3	6.8	8.66	96	97	—1	96	0
240	6 18.5	6.4	8.65	93	95	—2	93	0
255	7 4.6	6.0	8.64	91	92	—1	90	+1
270	7 14.8	5.5	8.63	89	90	—1	88	+1
285	8 0.9	5.0	8.61	85	87	—2	85	0
300	8 11.1	4.5	8.60	83	85	—2	83	0
315	8 21.3	4.0	8.59	81	82	—1	81	0
330	9 7.4	3.7	8.59	81	81	0	80	+1
345	9 17.5	3.5	8.58	79	80	—1	79	0
				Mean . .		±1.1	..	± 0.5

There seems to be no evidence of the term $n \sin (2v + \beta)$; in other words, the star appears to be a surface of revolution, one side being about four-fifths of the brightness of the other. It is also possible that the star may be elongated with axes in the ratio of four to five, but of equal brightness on all sides, and that its time of revolution is 20.32 days, or double the period commonly given. In this case there may be a slight difference in brightness at the alternate maxima or minima which has hitherto escaped detection, because not anticipated. From the second formula we may infer that the true maximum and minimum precede that adopted by Argelander, by the angular amount of 11.3°, or 7.6 hours. As this would only affect the light curve by about a fiftieth of a magnitude, it might readily escape detection. It will be noticed that in this case the interval from maximum to minimum is equal to that from minimum to maximum, instead of, as is generally the case, exceeding it. A more direct determination of the correction to the minimum may be found from the light curve, by comparing the times at which the light is equal.

In Table IV. are given the light in grades, the corresponding times

taken from the light curves, and the time of the minimum, assuming that it lies midway between them. This last column is found by adding to the second column $10^d\ 3.7^h$, or the period of the star; adding to this the third column, and dividing the result by two; finally subtracting the quotient from the period, $10^d\ 3.7^h$. The mean of this value, or 7.4, agrees closely with that given above.

TABLE IV. — MINIMUM OF ζ GEMINORUM.

Gr.	Increasing.	Decreasing.	Mean.
	d. h.	d. h.	h.
4.0	0 18.5	8 22.0	—5.6
5.0	1 12.0	8 0.5	—7.6
6.0	2 9.5	7 4.5	—6.9
7.0	3 13.5	6 2.0	—9.6

β Lyræ. Light curves of this star were given by Argelander in 1842, Astron. Nach., xix. 397, and in 1844, De stella β Lyræ variabili disquisitio. In 1859 he gave a more complete discussion of the problem.* He divided his previous observations into three periods, and derived a curve from each; concluding that they differed from each other only by their accidental errors, he gave a curve representing the entire series.

Oudemans † gives a light curve from his observations, reduced to the same system as that given in Argelander's second publication. This differs so little from the last system of Argelander that the same curve, for reduction to logarithms, has been used for both. In no case, within the limits used, would the difference of the logarithms exceed one or two hundredths.

In 1870, Schönfeld gave another curve, Astron. Nach., lxxv. 1. His grades represent a smaller variation in the light than Argelander's, and, like the latter, a grade is larger for the brighter stars than for the fainter, as in the case of *ζ Geminorum.* The relation of the grades to the logarithms of the light is shown in Table V. The columns have the same meaning as in Table II., except that three additional columns are given for the comparisons of Schönfeld.

* De stella β Lyræ variabili commentatio altera.

† Zweijaehrige Beobachtungen der meisten jetzt bekannten veraenderlichen Sterne. Verhand. Akad. Amsterdam, 1856.

TABLE V. — COMPARISON STARS FOR β LYRÆ.

Name.	Wolff.	Argelander.			Schönfeld.		
		Grades.	Curve.	$W-C$	Grades.	Curve.	$W-C$
γ Lyræ . .	8.89	12.7	8.87	+.02	15.0	8.88	+.01
μ Herculis .	8.79	..	..	..	13.0	8.80	—.01
ξ Herculis .	8.70	10.3	8.75	—.05	10.3	8.71	—.01
o Herculis . .	8.69	7.6	8.69	.00	7.8	8.65	+.04
ϵ Lyræ . .	[8.77]	4.9	8.60	[+.17]	4.2	8.58	[+.19]
ζ Lyræ . .	8.56	3.4	8.56	.00	2.9	8.56	.00
κ Lyræ . .	8.56	2.6	8.55	+.01	1.6	8.55	+.01

The estimate of ϵ *Lyræ* has not been included in drawing these curves. As the observations were made with the naked eye, in some cases aided by an opera-glass, ϵ and 5 *Lyræ* were treated as a single star. Wolff gives the logarithms of their light as 8.50 and 8.45. Their combined light would therefore equal 8.77, or nearly half a magnitude brighter than would be inferred from the estimate in grades, using the curves derived from the other stars. This may also be expressed by the statement that, together, they appear only a quarter of a magnitude brighter than either would alone, while a star of their combined brightness should appear about three quarters of a magnitude brighter than the separate components. It is possible that their proximity affected their measures by Wolff, but this seems less probable since they would be readily separated by the telescope of a Zöllner photometer. Evidently, ϵ *Lyræ* should not be used hereafter as a comparison star for this variable.

Table VI., like Table III., serves to compare the observations with theory. The first column gives the angle; the second, the corresponding time. Three sets of three columns each give the light in grades, in logarithms, and in percentages, for Argelander, Oudemans, and Schönfeld. Although the observations are not of equal value, it would be difficult to decide what weight should be given to each, and especially, to decide how large are the systematic errors to which each is subject. This last quantity should determine the weight, since the accidental errors are in a great measure eliminated by the smoothness of the light curves. Their mean, which is given in the next column, will accordingly be employed. The excess of the curve of each observer over the mean is given in the next three columns. An examination of the mean curve shows that it has two equal maxima symmetrically situated on each side of the point where $v = 180°$. The curve must therefore have the form $L = a + m \sin (v - 90°) + n \sin (2v - 90°)$.

The mean value of L or a is 81.1. When $v = 0°$, $L = a - m - n$; when $v = 180°$, $L = a + m - n$; $v = 90°$ or $270°$, gives $L = a + m - n$. Were there no accidental errors, either two of these three equations would determine m and n. After various trials the equation $L \doteq 81.1 + 4.1 \sin (v - 90°) + 20.0 \sin (2\,v - 90°)$ was found to give the most satisfactory results. The brightness computed by this formula, and the residuals found by subtracting them from the mean of the observed values, are given in the last two columns.

TABLE VI. — Variation in Light of β Lyræ.

v.	Time.	Argelander.			Oudemans.			Schönfeld.			Mean.	$A - M$	$O - M$	$S - M$	Comp.	$O - C$
		Gr.	Log.	Obs.	Gr.	Log.	Obs.	Gr.	Log.	Obs.						
°	d. h.															
0	0 0.0	3.4	8.56	49	4.0	8 57	49	3.6	8.57	51	50	−1	−1	+1	57	−7
15	0 12.9	5.0	8.60	56	5 0	8.60	52	4.4	8.58	58	55	+1	−3	+3	60	−5
30	1 1.8	9.2	8.71	72	8.0	8.67	62	9.2	8.71	71	68	+4	−6	+3	68	0
45	1 14.7	11.1	8.78	85	10.6	8.76	76	11 3	8.79	85	82	+3	−6	+3	78	+4
60	2 3.6	11.8	8.82	93	11 8	8.82	87	12 2	8.84	96	92	+1	−5	+4	89	+3
75	2 16.6	12.2	8.84	98	12 5	8.86	96	12.6	8.87	102	99	−1	−3	+3	97	+2
90	3 5.5	12.3	8.85	100	12.6	8.87	98	12.7	8.87	102	100	0	−2	+2	101	−1
105	3 18.4	12.1	8.84	98	12.4	8 85	93	12.5	8.86	100	97	+1	−4	+3	100	−3
120	4 7.3	11.8	8 82	93	11.7	8.81	85	11 9	8.82	91	90	+3	−5	+1	93	−3
135	4 20.2	11.2	8.79	87	10 7	8.76	76	10.9	8.77	81	81	+6	−5	0	84	−3
150	5 9.1	10.3	8.75	79	10 0	8.73	71	9.9	8.73	74	75	+4	−4	−1	75	0
165	5 22.0	8 8	8.69	69	9.2	8 71	68	9 1	8.70	69	69	0	−1	0	68	+1
180	6 10.9	8.6	8.69	69	9.1	8.70	66	8.9	8 70	69	68	+1	−2	+1	65	+3
195	6 23.8	9.4	8.71	72	9.5	8.72	69	9.3	8.71	71	71	+1	−2	0	68	+3
210	7 12.7	10.8	8.77	83	10.7	8.76	76	10.2	8.74	76	78	+5	−2	−2	75	+3
225	8 1.6	11 6	8.81	91	11.8	8.82	87	11.2	8.79	85	88	+3	−1	−3	84	+4
240	8 14.5	12.1	8.83	96	12.4	8.85	93	12.0	8.83	93	94	+2	−1	−1	93	+1
255	9 3.5	12.3	8.85	100	12 8	8.88	100	12.4	8 85	98	99	+1	+1	−1	100	−1
270	9 16.4	12.4	8.85	100	12.9	8.88	100	12.4	8.85	98	99	+1	+1	−1	101	−2
285	10 5.3	12.2	8.84	98	12.8	8 88	100	12.3	8.85	98	99	−1	+1	−1	97	+2
300	10 18 2	11.7	8.81	91	12.4	8.85	93	11.9	8.82	91	92	−1	+1	−1	89	+3
315	11 7.1	10.9	8.77	83	11.4	8.80	83	10.8	8.77	81	82	+1	+1	−1	78	+4
330	11 20.0	8.4	8.68	68	9.0	8.70	6[illegible]	8.0	8.67	65	66	+2	0	−1	68	−2
345	12 8.9	4.0	8.58	54	4.7	8.59	51	4.2	8.58	53	53	+1	−2	0	60	−7
										Mean	. .	±1.9	±2.5	±1.5	..	±2.8

These residuals are much larger than in the case of ζ *Geminorum;* but this is to be expected, since the variations in light are greater. Evidently, if the changes were small, any two smooth curves would agree closely. Their average value amounts to about .04 of a magnitude, and their greatest value does not exceed the greatest difference of each of the observed curves from the others. The greatest errors of observation are those of the light of the comparison stars. The residuals near the principal minimum may be greatly reduced if the fainter comparison stars are assumed too faint, with a corresponding change in the value of the fainter grades. The rejection of ϵ *Lyræ* in Table V., while its effect on Table VI. cannot be eliminated, may account for this apparent error. An increase in the logarithm of the

light by about .01 for grades 11 to 12 would reduce the residuals corresponding to $v = 45°$, $60°$, $210°$, $225°$, $300°$, and $315°$. The residuals for $v = 120°$ and $135°$ would, however, be increased. These changes are not to be recommended unless indicated by future photometric measures of the comparison stars.

η Aquilæ. Argelander gave a light curve of this star in 1842, Astron. Nach., xix. 399, based upon 174 observations taken by himself and by Heis. In 1857, he gave a second curve dependent on 411 of his own observations, Astron. Nach., xlv. 97. In Table VII. the light of the comparison stars are given in grades and in logarithms, according to Wolff. The columns have the same meaning as in Table V. As *ν Aquilæ* was not observed by M. Wolff, it is unavoidably excluded from the comparison.

TABLE VII. — COMPARISON STARS FOR η AQUILÆ.

Name.	Wolff.	Grades.	Curve.	$W-C$	Grades.	Curve.	$W-C$
δ Aquilæ . .	8.85	13.3	8 85	.00	12.9	8 86	+.01
β Aquilæ . .	8.74	8.0	8.72	+.02	8.1	8.71	—.03
ε Aquilæ . .	8 63	6.0	8.66	—.03	6.1	8.64	+.01
ι Aquilæ . .	8.57	3.0	8.57	.00	3.0	8.55	+.02
μ Aquilæ . .	8.43	—1.4	8.43	.00	—0.6	8.44	—.01
ν Aquilæ . .	...	—2.4	...	..	—1.8	...	..

The first seven columns of Table VIII. have the same meaning as in Table III. and give the values of v, of the time, of the light in grades, in logarithms, and in percentages, a computed value, and the residuals from this, or the fifth column minus the sixth. The computation is effected by the formula $L = 73.6 + 20.0 \sin (v - 60°) + 6.0 \sin (2v - 120°)$. The last four columns give the light in grades, logarithms, and percentages, and the residuals according to the second curve of Argelander. The same theoretical formula is used in this case, adding 1 so that the positive and negative residuals shall be nearly equal. The light in this case, $L' = L + 1 = 74.6 + 20.0 \sin (v - 60°) + 6.0 \sin (2v - 120°)$.

δ Cephei. Argelander has given a light curve of this star in the Astron. Nach., xix. 395. Curves are also given by Oudemans in the paper cited above, and by Schönfeld in Astron. Nach., lxxv. 14. The relation of grades to logarithms is given in Table IX., for Argelander and Schönfeld. Oudemans has already reduced his scale to that of Argelander. Unfortunately, Wolff only measured those of the five

TABLE VIII. — VARIATION IN LIGHT OF η AQUILÆ.

v.	Time. d.	h.	Gr.	Log.	Obs.	Comp.	$O-C$	Gr.	Log.	Obs.	$O-C'$
0°	0	0.0	1.2	8.51	50	51	−1	2.1	8.52	54	+2
15	0	7.2	1.7	8.53	52	54	−2	2.4	8.53	55	0
30	0	14.4	3.2	8.57	58	58	0	3.5	8.56	59	0
45	0	21.5	4.8	8.63	66	65	+1	4.8	8.60	65	−1
60	1	4.7	6.3	8.67	72	74	−2	6.2	8.65	72	−3
75	1	11.9	8.1	8.72	81	82	−1	7.8	8.70	81	−2
90	1	19.1	9.6	8.76	89	89	0	9.5	8.75	91	+1
105	2	2.3	10.9	8.79	96	94	+2	10.6	8.78	98	+3
120	2	9.4	11.4	8.81	100	96	+4	10.9	8.79	100	+3
135	2	16.6	10.9	8.79	96	96	0	10.5	8.78	98	+1
150	2	23.8	10.1	8.77	91	94	−3	9.8	8.76	93	−2
165	3	7.0	9.2	8.75	87	90	−3	9.0	8.74	89	−2
180	3	14.2	8.5	8.74	85	86	−1	8.2	8.71	83	−4
195	3	21.3	8.0	8.72	81	82	−1	8.0	8.70	81	−2
210	4	4.5	7.6	8.71	79	78	+1	7.8	8.70	81	+2
225	4	11.7	7.2	8.70	78	76	+2	7.2	8.68	78	+1
240	4	18.9	6.8	8.69	76	74	+2	6.6	8.66	74	−1
255	5	2.0	6.4	8.67	72	71	+1	6.0	8.64	71	−1
270	5	9.2	5.8	8.65	69	69	0	5.4	8.62	68	−2
285	5	16.4	4.7	8.62	65	66	−1	4.9	8.61	66	−1
300	5	23.8	3.7	8.59	60	62	−2	4.2	8.58	62	−1
315	6	6.7	2.9	8.56	56	57	−1	3.6	8.57	60	+2
330	6	13.9	2.1	8.54	54	54	0	3.0	8.55	58	+3
345	6	21.1	1.5	8.52	51	51	0	2.4	8.53	55	+3
					Mean	. .	±1.3				±1.8

comparison stars of δ *Cephei*, and the logarithms of two of these he found differed by only one hundredth of a unit. Accordingly, we can do no better than to draw a straight line nearly through the points designated by these stars, or assume that the value of a grade is constant. The columns of Table IX. have the same meaning as the corresponding columns of the previous similar tables.

TABLE IX. — COMPARISON STARS FOR δ CEPHEI.

Name.	Wolff.	Grades.	Curve.	$W-C$	Grades.	Curve.	$W-C$
ζ Cephei . .	8.84	11.4	8.84	.00	12.4	8.86	−.02
ι Cephei . .	8.83	10.8	8.82	+.01	10.9	8.82	+.01
7 Lacertæ .	. . .	7.1	. . .	. .	6.6	.	. .
ξ Cephei . .	. . .	3.0	. . .	. .	. . .	. .	. .
ϵ Cephei . .	8.53	2.0	8.53	.00	1.9	8.53	00

Table X. compares the various light curves with theory. The columns have the same meaning as those of Table V. The theoretical values are computed by the formula, $L = 72.1 + 20.0 \sin (v - 45°) + 7.0 \sin (2v - 120°)$

TABLE X. — VARIATION IN LIGHT OF δ CEPHEI.

v.	Time.	Argelander.			Oudemans.			Schönfeld.			Mean.	$A - M$	$O - M$	$S - M$	Comp.	$M - C$
		Gr.	Log.	Obs.	Gr.	Log.	Obs.	Gr.	Log.	Obs.						
°	d. h.															
0	0 0.0	2.8	8.56	55	3.4	8.58	55	3.0	8.56	58	56	−1	−1	+2	52	+4
15	0 5.4	3.0	8.56	55	3.2	8.57	54	3.1	8.57	59	56	−1	−2	+3	55	+1
30	0 10.7	3.5	8.57	56	3.3	8 57	54	3.5	8.58	60	57	−1	−3	+3	61	−4
45	0 16.1	4.7	8.62	63	5.3	8.64	63	4.6	8.61	65	64	−1	−1	+1	69	−5
60	0 21.5	6.5	8.68	72	7.5	8.71	74	6 6	8.68	76	74	−2	0	+2	77	−3
75	1 2 8	8 4	8.74	83	10 3	8.81	93	8.7	8.75	89	88	−5	+5	+1	86	+2
90	1 8.2	9 9	8 79	93	11.1	8.83	98	10.0	8.79	98	96	−3	+2	+2	92	+4
105	1 13.6	10.7	8.82	100	11.2	8.84	100	10.4	8.80	100	100	0	0	0	96	+4
120	1 19.0	10.1	8.80	96	10.9	8 83	98	10.0	8.79	98	97	−1	+1	+1	98	−1
135	2 0 3	9.0	8.76	87	10.4	8.81	93	9.3	8.76	91	90	−3	+3	+1	96	−6
150	2 5.7	8.5	8.75	85	9.6	8.78	87	8.6	8 74	87	86	−1	+1	+1	91	−5
165	2 11.1	8.4	8.74	83	8.6	8.75	81	8.0	8 72	83	82	+1	−1	+1	86	−4
180	2 16.4	8.3	8.74	83	7.7	8.72	76	7.8	8.72	83	81	+2	−5	+2	80	+1
195	2 21.8	7.8	8.72	79	6.8	8.69	71	7.6	8.71	81	77	+2	−6	+4	75	+2
210	3 3.2	7.1	8.70	76	6.2	8.67	68	6 8	8.69	78	74	+2	−6	+4	71	+3
225	3 8 5	6.3	8.67	71	5.6	8.65	65	6.2	8.67	74	70	+1	−5	+4	69	+1
240	3 13 9	5.6	8.65	68	5.0	8.63	62	5.6	8.65	71	67	+1	−5	+4	67	0
255	3 19 3	5.2	8.64	66	4.4	8.61	59	5.1	8.63	68	64	+2	−5	+4	66	−2
270	4 0.6	4.7	8 62	63	4.0	8.60	58	4.7	8.62	66	62	+1	−4	+4	64	−2
285	4 6.0	4.3	8.61	62	3 7	8.59	56	4.3	8.60	63	60	+2	−4	+3	62	−2
300	4 11.4	3.9	8.59	59	3.6	8 58	55	3.9	8 59	62	59	0	−4	+3	59	0
315	4 16.7	3.4	8 58	58	3.6	8.58	55	3 6	8.58	60	58	0	−3	+2	56	+2
330	4 22.1	3.2	8.57	56	3.6	8.58	55	3.3	8.57	59	57	−1	−2	+2	53	+4
345	5 3.5	2.9	8.56	55	3.5	8.58	55	3.1	8 57	59	57	−1	−1	+3	51	+6
										Mean . .		±1.4	±2.9	±2.8	.:	±2 8

These residuals are not large, considering the differences between the different observed values. There is, however, a curious alternation of the positive and negative signs. As a similar alternation appears in some of the other residuals, it is important to compare them to see if they can be shown to follow any law. There appear to be three maxima and three minima, or the variation repeats itself at intervals of about 120°. We should then exaggerate this effect by adding each set of the three residuals differing by 120°; that is, the residuals corresponding to 0°, 120°, and 240°, to 15°, 135°, and 255°, etc. This is done in Table XI., in which the first value of v, in each set, is given in the first column, and the sums of the three residuals for the four stars are given in the second, third, seventh, and eleventh columns. The residuals of ζ *Geminorum* are so small that we should expect no evidence of systematic error. In the other three cases marked variations are shown. In each case there are only two changes of sign, while there should be on the average four if the variations were accidental. The residuals of β *Lyræ* are well represented by subtracting from the computed value 3 cos 3 v. The residuals which then remain are given in columns four, five, and six. Their average value is 1.6 instead of 2.8, or they have been reduced nearly one half. The residuals of η *Aquilæ*, in like manner, leave 1.1 instead

of 1.8, by subtracting the term 3 sin $(3v - 45°)$. Those of δ *Cephei* become 1.4 instead of 2.8, if we subtract 4 sin $3v$.

TABLE XI. — TERMS INVOLVING $3v$.

v.	ζ Gemine.	β Lyræ				η Aquilæ.				δ Cephei.			
		Sum.	Residuals.			Sum.	Residuals.			Sum.	Residuals.		
0°	0	−9	−4	0	+4	+5	0	+1	−3	+ 3	+4	−1	0
15	0	−9	−3	−1	+1	0	0	+1	−1	− 7	+4	−3	+1
30	−2	−2	0	0	−2	−4	+2	0	0	−11	0	−1	+2
45	+1	+7	+2	−1	0	−4	+2	+1	+2	−11	−2	−1	+1
60	+1	+9	0	0	0	−8	−1	−2	+1	− 2	−3	+1	0
75	−1	+9	0	+1	+2	−2	−2	−2	+2	+ 6	−1	−1	−1
90	0	0	−1	+3	−2	+6	−1	0	+1	+11	0	−1	0
105	−1	−6	−1	+6	−5	+7	0	+2	0	+11	+1	−2	+3
Mean. .		±1.6				±1.1				±1.4			

No natural explanation can be offered for such terms, and the reduction might be thought accidental did it not occur in so many different curves. A careful distinction must be made between these terms and those which might be assumed empirically, since their form is clearly pointed out by the residuals. If we tried to represent the residuals by a function of $4v$, we should soon see that the effect was wholly different, nor would any values of the arbitrary constants in this case materially reduce the residuals.

Neglecting these last terms, as their reality may be questioned, we may write the equations of the four stars under each other thus: —

ζ Geminorum, $L = 89.6 + 10.2 \sin (v - 11.3°)$
β Lyræ, $L = 81.1 + 4.1 \sin (v - 90°) + 20.0 \sin (2v - 90°)$
η Aquilæ, $L = 74.6 + 20.0 \sin (v - 60°) + 6.0 \sin (2v - 120°)$
δ Cephei, $L = 72.1 + 20.0 \sin (v - 45°) + 7.0 \sin (2v - 120°)$

To compare them, it will be convenient to make the mean brightness equal to unity in all cases, or to divide by a the equation $L = a + m \sin (v + \alpha) + n \sin (2v + \beta)$. Instead of making $v = 0$, when the light is a minimum, it will also be better to take as the starting-point the position in which the shorter axis of the star is turned towards the observer. If $v' = v + \gamma$, we may write $L' = 1 + m' \sin (v' + \alpha') + n' \cos 2v'$. The various values of these constants are given in Table XII., which contains in successive columns the name of the star, the value of γ, of α', of m', and of n'. Independently of the form of the star, its light would vary, owing to the unequal bright-

ness of the two sides from $1 + m'$ to $1 - m'$. The brightness of the darker side would therefore equal $\frac{1 - m'}{1 + m'}$ times that of the brighter. In like manner, if the surface was uniformly bright, the variation in area of the disk, or the length of the shorter axis in terms of the longer, would be $\frac{1 - n'}{1 + n'}$. These quantities are given in the sixth and seventh columns. The last two columns give the average residuals in percentages before and after applying the terms which are functions of $3\,v$.

TABLE XII. — COMPARISON OF LIGHT CURVES.

Name.	γ	a'	m'	n'	$\frac{1-m'}{1+m'}$	$\frac{1-n'}{1+n'}$	Av. Resid.	Av. Resid.
	$^\circ$	$^\circ$						
ζ Geminorum	$-$ 11.3	. .	+.114	. .	0.80	. .	0.5	. .
β Lyræ . . .	$-$ 90.0	0	+.051	+.247	0.90	0.60	2.8	1.6
η Aquilæ . .	$-$105.0	+45	+.268	+.080	0.58	0.85	1.8	1.1
δ Cephei . .	$-$105.0	+60	+.277	+.097	0.57	0.82	2.8	1.4

From the column $\frac{1 - m'}{1 + m'}$ we see that in every case the darker side is more than half as bright as the other, and that the difference in the case of β *Lyræ* amounts only to ten per cent. In other words, if this effect is due to spots, we must conclude that they cover only one-tenth of the hemisphere in the case of β *Lyræ*, and about two-fifths in the cases of η *Aquilæ* and δ *Cephei*. The next column also shows that β *Lyræ* is much elongated, the ratio of its axes being as five to three, while the two stars following have this ratio about as six to five.

The dark portion of β *Lyræ* is at one of the ends, since $\alpha' = 0^\circ$ for this star; it appears also to be symmetrically situated as regards the longer axis. The dark portions, both of η *Aquilæ* and of δ *Cephei*, are placed somewhat preceding an end, that is, they are turned towards the observer before the end has been directed to him. For this reason the time from minimum to maximum is greater than that from maximum to minimum. This is probably a general law of stars of this class, as it has been noticed in several other instances.

One source of systematic error has been disregarded in the above comparison of observation with theory. In the value of L' the term $m' \sin (v' + \alpha')$ may be regarded as the measure of the effect of the difference in brightness of the two sides, and $n' \cos 2\,v'$ as due to the form of the body. Their combined effect, however, would not strictly equal their sum, but would be found by adding each to unity and

taking the products of these sums. The actual light would equal $(1 + m' \sin (v' + a')) (1 + n' \cos 2 v') = 1 + m' \sin (v' + a) + n' \cos 2 v' + m'n' \sin (v' + a) \cos 2 v'$. The value of L' given above is then subject to the systematic error of $m'n' \sin (v' + a) \cos 2 v'$. The maximum value of this would equal $m'n'$, and it would generally be much less. The maximum value for β *Lyræ* would be about 1 per cent; for η *Aquilæ*, 2; and for δ *Cephei*, 2.6 per cent. If the star underwent much greater change of light, it might be necessary to take this term into account; but in the present case it does not seem to sensibly affect the average value of the residuals.

Various attempts have been made to determine the light curve of β *Lyræ* photometrically. The observations of Zöllner and Wolff are reduced according to the same method in the photometric work of the latter, p. 110. The accuracy of the results does not make this a promising method of determining the light curve, unless the number of observations is greatly increased. The maxima and minima were also determined at the Harvard College Observatory.* Calling the light at either maximum 100, that at the two minima would be 55.8 and 64.7, which agrees very closely with that given by computation, if we neglect the term $3 \sin 3 v$.

One great advantage of the study of the stars by physical instruments, such as the spectroscope and photometer, is that some clew is given to certain laws, for our knowledge of which we must otherwise depend on theoretical considerations alone. While the conclusions to be drawn from micrometric measurements are in general much more precise, and the effect of the errors can be more certainly computed, they fail entirely to aid us in studying such laws as those here considered. For example, the present investigation serves to study the following important problem in cosmogony, to which micrometric measures contribute nothing, and which can otherwise only be examined from the standpoint of theory. If we admit a common origin to the stars of the Milky Way, a general coincidence in their axes of rotation seems not improbable, especially as such an approximate coincidence occurs in the members of the solar system. If the coincidence was exact, the direction must be that of the poles of the Sun, or, approximately, that of the pole of the ecliptic. On the other hand, since the stars of the Milky Way are supposed to be arranged in the general form of a flattened disk, we should more naturally expect that the axes of rotation would be symmetrically situated with regard

* Annals, xi. 135.

to it, or would coincide with its shortest dimension. According to this theory, then, the axes of rotation would be directed towards the poles of the Milky Way. If now we suppose that a great number of variable stars, of the form described above and rotating around parallel axes, were distributed over the heavens, it is evident that those seen in the direction of their axes would not appear to vary, since as they turned they would always present the same portion of their surfaces to the observer. Those at right angles to this direction would show the greatest variation, and, other things being equal, would appear to be more numerous since they would be more likely to be detected. If then the axes are coincident, we should expect that most of these variable stars would lie along the arc of a great circle whose pole would coincide with their axes of rotation. An inspection of a plot of the stars of Class IV. showed that they agreed closely with a great circle whose pole is in R. A. 13^h and Dec. $+20°$. To compare these stars in this and in other respects, they are arranged in the order of their periods in Table XIII. They are divided into three sections; first, those known to be of the fifth class; secondly, those of the fourth class, including all of a shorter period than *β Lyræ;* thirdly, the remaining variables of longer period, whose position in Class IV. may be open to question. The first column gives the name of the star, and the second its period in days. The distance from the great circle whose pole is in R.A. 13^h and Dec. $+20°$ is given in the third column. It was found by measurement on a globe, instead of by calculation, and is not therefore exact to the nearest degree.

In measuring the stars of the fifth class at the Harvard College Observatory, much difficulty was experienced from the absence of adjacent comparison stars. Stars of the fourth class, on the other hand, have, in almost all cases, stars near them. An unprejudiced comparison is made in the next two columns, by giving the magnitude and distance, in minutes, of the nearest star of the Durchmusterung. The lines for the southern stars are therefore left blank. If the stars of the fourth class lie near the Milky Way, we should expect an increased number of companions due to this cause. Accordingly, a count has been made of the Durchmusterung stars in a square degree, in which each star is contained. This area is defined as the portion of the Durchmusterung zone in which the star is situated, having an average length of one degree, one half preceding, the other half following, the variable. The results are given in the sixth column. If these stars were connected with the variables, we might expect that they would lie, approximately, in a plane at right angles to the axes of rota-

TABLE XIII. — COMPARISON OF VARIABLE STARS.

Name.	Period.	Dist.	Mag.	Dist.	No. Stars.	Ang.	Birm.
Class V.							
		°		′		°	
δ Libræ	2.32	+51	..	..	..	..	..
— Cephei	2.49	+11	9.5	5.1	20	..	..
β Persei	2.87	—24	8.8	7.5	15	..	55
U Coronæ	3.45	+58	9.4	11.5	7	..	..
λ Tauri	3.95	—41	9.5	17.5	5	..	..
S Cancri	9.48	+25	9.1	11.6	16	..	..
Mean		±35°	..	10′.6	12.6	..	..
Class IV. — Short Periods.							
R Muscæ	0.89	0	..	..	..	..	..
T Triang. Austr.	1.00	+ 3	..	..	..	..	..
— Sagittarii	2.42	— 1	..	..	..	..	..
S Monocerotis	3.40	— 4	9.4	1.0	29	—31	..
R Triang. Austr.	3.40	+ 1	..	..	..	..	..
N Velorum	4.25	+ 1	..	..	..	..	..
δ Cephei	5.37	— 5	7.5	1.5	31	—33	..
S Coron. Austr.	6.20	— 9	..	..	..	..	..
U Sagittarii	6.75	+ 2	..	..	..	..	445
X Sagittarii	7.01	+ 8	..	..	..	..	..
η Aquilæ	7.18	— 9	9.2	3.2	17	+20	..
W Sagittarii	7.59	+ 3	..	..	..	..	..
κ Pavonis	9.10	—16	..	..	..	..	..
ζ Geminorum	10.16	+ 5	8.5	1.8	33	+ 1	..
β Lyræ	12.91	+14	8.5	1.5	34	—20	..
Mean		±5°	..	1′.7	28.8	±21°	..
Class IV. — Long Periods.							
W Virginis	17.27	+66	..	..	..	..	284
T Monocerotis	27.00	— 8	9.5	6.1	23	0	..
l Carinæ	31.25	— 1	..	..	..	..	..
u Herculis	38.50	+34	9.4	5.0	15	+79	405
U Monocerotis	46.00	+ 2	..	..	..	..	..
R Lyræ	46.00	+16	7.1	9.7	17	—78	474
R Coron. Austr.	54.00	— 9	..	..	..	..	..
S Vulpeculæ	67.50	0	9.5	5.2	20	+53	517
R Sagittæ	70.42	— 9	9.3	5.0	26	+76	540
R Scuti	71.10	+ 3	..	..	..	..	462
Mean		±13°	..	6′.2	20.2	±57°	..

tion, since the planes of revolution of the planets do not differ greatly from the solar equator. Moreover, if the elongation of the variable was caused by one or more disturbing bodies, we should expect that they would lie in this plane. Of course, the present distance of these

companions is far too great to sensibly affect the variables, but other nearer objects may lie in the same plane. The approximate position angle of the companion was computed from its Durchmusterung place. The position angle of the pole of the variable stars was measured by a protractor, laid upon the globe over the position of the variable star, and stretching a thread to the pole. Each of these determinations is liable to an error of some degrees, but the results which are given in column seven are sufficiently exact for our present purposes. Some of these stars are red, and when they are contained in the Catalogue of Birmingham * their numbers are given in the last column.

The numbers of the third column show that the stars of the fifth class are not concentrated in the assumed plane. If uniformly distributed all over the heavens, their average distance should be about 30°, since one half of each hemisphere is contained in a zone of this width. In the short-period stars of the fourth class, however, the agreement is most remarkable. None have yet been found more distant than 16° from the circle, and with two exceptions none are more distant than 10°. There is only one chance in four that a given star should lie within 15° of a given great circle, and about one in six that it should lie within 9° of it. Evidently the chances would be many millions to one against the observed arrangement being accidental. As an argument in favor of the parallelisms of the axes, this distribution of the stars fails by proving too much. We should expect, if the axes were parallel, to find nearly as many stars between 10° and 20°, as between 0° and 10°, since the variation would be a function of the cosines of these angles. If the axes were not exactly coincident, we should find the stars still more widely distributed.

Of course it is possible that the distribution of these stars may be partly due to the parallelism of their axes of rotation. But we have shown that the latter cause is insufficient. Since then we must assume an arrangement of the stars approximately in a plane, we cannot be sure that their apparent distribution is not wholly due to it, and the evidence in favor of parallelism of their axes is much weakened.

It is a little singular that this plane appears to pass through the Sun. We should expect that while the more distant stars might lie upon a great circle, the nearer, and therefore presumably the brighter, stars, would lie on the opposite side of it from the Sun. As, however, the positive and negative signs are nearly equally distributed, we must

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* Trans. Roy. Inst. Acad., xxvi. 249.

infer that the distance of the Sun from the plane of these stars is small compared with its distance from them. If the stars lay exactly in one plane we might infer their distances from the Sun from these residuals. As the residuals of the brighter stars show no systematic arrangement, it seems probable that the variables of the fourth class lie nearly, but not exactly, in a plane. This plane approaches that of the Milky Way, but does not coincide with it. The pole of the latter is nearly in R. A 12^h 40^m and Dec. $+28°$. Evidently the residuals in column three would be greatly increased if we moved the pole from its assumed position of R. A. 13^h and Dec. $+20°$, by more than $10°$ to the pole of the Milky Way. The position of the Milky Way, as given in the "Atlas Coelestis Novus" of Heis, agrees, however, more closely with the plane of the variable stars.

It is not certain whether the stars of longer period given in the third section of Table XII. should be included with those of the fourth class of variables. With two exceptions, *W Virginis* and *u Herculis*, they lie near the plane of the others.

The total number of stars in the Durchmusterung north of the equator is 315,048. Since the area of the hemisphere is 20,626 square degrees, this corresponds to 15.3 stars per degree, or an area of 236 square minutes to each star. A circle having a radius of $8'.7$ would have an area equal to this. If, then, a circle having this radius is described around any star as a centre, it will be an even chance that another star will be contained within it, provided that the presence of the second star is no way affected by that of the first. For circles of other radii the chances will vary as the squares of the radii, or as the areas. We know from the existence of clusters and multiple stars that one star is not without influence on the presence of another, and that this effect may extend to some distance, as is shown in the Pleiades and in Præsepe. This principle may still be used in comparing different classes of stars, although the distance $8'.7$ should be diminished. It is, therefore, surprising that the average distance of the companions of stars of the fifth class is as great as $10'.6$, especially as two of them, *S Cancri* and *λ Tauri*, lie near the large clusters Praesepe and the Hyades, where the average intervals between the stars is much less. A circle of radius $10'.6$ has only two-thirds the area of that of $8'.7$, hence these companions are only two-thirds as thickly placed as the stars in other parts of the heavens. This effect extends to the square degree, as is shown in the sixth column. It appears to be probable that there is no physical connection of these stars with the variables, and that their sparseness is due to their distance from the Milky Way.

Passing now to the second part of the table, we find a wholly different condition of things. Every star has a companion near it at an average distance of only 1′.7, or these stars are twenty-six times as thickly placed as in the rest of the heavens, since $8.7^2 : 1.7^2 = 26 : 1$. This effect is partly due to the surrounding square degree, which contains nearly double the average number of stars. Only a small part of this effect may, however, be thus explained. We may, therefore, infer that there is a physical connection between these variables and their companions, or that they are at nearly the same distance from the Sun, and not optically double. The singular character of these stars renders them interesting objects for the measurement of parallax. This is especially the case with those of very short period, since from the rapidity of the changes we might infer that they were really small, and therefore near. Now an observer would be very likely to select the companions as points to measure from, since their distances are much greater than that separating the components of most stars which are binary, or are supposed to be physically connected. A measurable parallax might thus escape detection.

The stars of longer period occupy an intermediate position as regards the distances of the components, and the number of stars in the square degree.

If the direction of the components depended wholly on chance, we should find that they would differ, on the average, from that given by any theory, by about 45°. It therefore seems scarcely probable that, in each of the five cases, a chance distribution would give the angle less than 45°, for the stars of short period. The uncertainties in the measurements would in general increase the discrepancies, so that it is to be expected that a more accurate determination would diminish the mean value, although it would doubtless alter the separate results by many degrees. The position of the components of the stars of the fifth class has not been determined, as it seems very improbable that they have any physical connection. The stars of long period, with one exception, give results which do not agree at all with theory. Some more precise test of the class to which these variables should be assigned, is therefore much needed. They are distinguished from many of the stars of the second class only by the length of their period, no other known variables having a period less than that of *R Vulpeculæ*, or 137 days. Stars of Class II. have banded spectra, and are of a red color. This suggested a test dependent upon observations already made. The last column shows what stars have been regarded as red, and may, therefore, in some cases belong to Class II. The only

star of Class V. given in Birmingham's Catalogue is *β Persei*, and many observers may be surprised that this should have been called a red star. It is remarkable that but one star of short period, *U Sagittarii*, is called red. On the other hand, six of the variables of long period are given in the catalogue, including all of those which have shown marked discrepancies. Excluding these disposes of the large deviations, 66° and 34°, in column three; and we find no star more distant than 16° from the assumed plane in which the variables lie. Again, the large discrepancies of the last column but one are removed, and *T Monocerotis* probably placed with the variables of the fourth class. This view is confirmed by the light curve given by Schönfeld, page 32 of the catalogue cited above, which shows that in the form of its variations this star closely resembles *η Aquilæ* and *δ Cephei*. Another reason for excluding *W Virginis* and the last four stars of the list is, that their light is variable at their maxima, and in four of the five cases at their minima. This frequently happens with stars of Class II., but would not be readily explained in stars of Class IV.

The Uranometria Argentina adds *U Monocerotis* to the list of red stars. All stars whose period lies between 32 and 72 days have, therefore, been called red, except *R Coronæ Australis*. This star is so faint that its color might well have been overlooked.

A further discussion would have been made of *T Monocerotis*, but no means exist for converting into light ratios the scale of magnitudes of its light curve. As the brightness of the comparison stars are not given, we have no means of knowing whether a tenth of a magnitude corresponds to the same light ratio when this star is faint, as when it is bright. A preliminary trial showed that the maximum appeared to occur more suddenly, and the minimum more slowly, than theory would indicate. The large range of variation of this star renders it well suited for study, and the same may be said of some others of the list, as a slight increase in the difference between the maximum and minimum greatly increases the severity of the test the light curve offers to theory.

The system which appears to govern the position of the companions to these stars suggests an investigation which might lead to important results. The planes of the orbits of the binary stars are defined by their inclination and the position angle of the node. Since we cannot determine micrometrically the direction in which the orbit is inclined, we can only say that the pole of this orbit lies in one of two places. Should any law be discovered, we might then decide for any particular star what sign should be given to the inclination, and also whether the

motion was direct or retrograde. It might also help to determine the amount of the inclination when the latter is not large enough to be determined precisely by micrometric measurements. Such a law would render an important aid to the study of the orbits of the dark companions of stars of the fifth class. They would afford a check on the observed inclination, and would define the position angle of the major axis of the orbit, which is now wholly indeterminate. An inspection of the orbits of the binaries fails to show any law, but it is possible that this might be brought out by a more careful examination, as has been done with the proper motion of the stars. The conclusion regarding the motion of the Sun in space is liable to large error, in case systematic errors exist in the catalogues on which the positions of the stars depend. Such an error in Bradley might greatly change the conclusion generally accepted. The orbits of the binaries, on the other hand, are wholly independent of each other, and there is little danger of a systematic error affecting all.

The elegant method of Argelander for determining the light curve of the variable stars leaves little to be desired as a means of determining their periods and the times of their minima. Its simplicity, and the need of no instrument but a telescope powerful enough to show the variable, are strong arguments in its favor when comparing it with the best photometric methods. If, however, we wish to determine the true light curve, the following sources of error become perceptible. As the comparison stars are selected from the immediate neighborhood of the variable, they are few in number; and if any one of them proves to be itself variable, the errors introduced are large. It is difficult to obtain independent estimates, since there is but little range of choice in the star to be selected at any given time. Much skill is required on the part of the observer to make a grade the same when the variable is bright as when faint, to make it the same on different nights, and to make the interval of two grades double that of one. In reducing the light to logarithms, it appears to be impossible to render the errors of the measures of the comparison stars as small as those of the light curves. The comparisons given above show that the errors of the measurements of the comparison stars probably exceed those from all other sources combined.

Three methods may be used for determining the brightness of the stars without a photometer. First, the observer may keep a certain scale in mind, and by it estimate the light of the stars in tenths of a magnitude. He should first estimate several known stars, and compare his result with their true brightness, so as to apply mentally to his

scale proper corrections for the effect of haze, moonlight, etc. He may also observe a large number of known stars, and afterwards reduce his scale for the evening from a discussion of their light.

In the second method, which is that of Argelander, the observer selects a comparison star of very nearly the same light as the star to be measured, and estimates the difference in grades, a grade being a small interval nearly equivalent to a tenth of a magnitude. A discussion of all the observations serves to determine the intervals in grades between the comparison stars. The value of one grade is then determined from photometric measures of the comparison stars. According to the third method, the observer selects two comparison stars, one a little brighter, the other a little fainter, than the star to be observed, and estimates its difference in magnitudes from the brighter component, with the difference between the two comparison stars. The first of these methods is the most rapid, and is well adapted to zone observations, or to any work with a meridian instrument. More skill is, however, required on the part of the observer than by either of the other methods. Besides being able to judge of small intervals of brightness, as in the other methods, he must be able to prevent any changes from taking place in his scale, at least during a single evening. The second method requires less skill, since the observer must merely keep the values of his grades constant; but in the third method even this is not needed. It is, therefore, probably the most exact, when the results are to be reduced by photometric measures of the comparison stars. The three methods are directly comparable with those which may be used in estimating linear measures. We may estimate the length of a bar directly in inches, or its excess in inches over a similar bar of known length; or, thirdly, we may compare it with two bars, one a little longer, the other a little shorter, and estimate its relative length compared with them. It can hardly be doubted that the last of these methods would give the most accurate results. When applied to the stars, the third method has also an advantage in reducing the accidental errors of the photometric measures, since the comparison is made with two stars instead of one.

The light curve of a variable may then be determined as follows: — Select as comparison stars all those of nearly the brightness of the variable, and not too far distant, excepting any which may be thought to be variable, to differ from the variable in color, or which are near other stars. Photometric measures should be obtained, during the period over which the observations of the variable extend, of all of those stars which are used. Each star should be measured in turn under precisely the same conditions, by a Zöllner photometer or other

instrument, and this should be repeated on several evenings. The relative light will thus be obtained with great accuracy, as the same errors will be likely to affect them all. If this cannot be done, the Uranometria Argentina, with the measures now in progress at the Harvard College Observatory, will give the brightness of all the naked-eye stars, with an error probably less than a tenth of a magnitude.

The light of the variable would be found by selecting two comparison stars, one a little brighter, the other a little fainter than it, and comparing the interval between the variable and the brighter, with that between the two comparison stars, which may be assumed equal to 10. Thus, *a* 4 *b* will denote that the interval between the bright comparison star *a* and the variable is estimated at only four-tenths of that between the two comparison stars. Of course the time of each comparison must be recorded. This measure should be repeated with different pairs of comparison stars. Thus, if *a* and *b* are brighter and *c* and *d* fainter than the variable, we may compare the latter with *ac*, *ad*, *bc*, and *bd*. In like manner, with six comparison stars we may obtain nine independent measures. The reduction is very simple, since it is useless to carry the estimates beyond tenths of a magnitude.

The above paper has suggested several researches of importance, and which are accordingly placed together below: —

1. Determination of the light curves of any of the variables of short period, except β *Persei*, ζ *Geminorum*, β *Lyræ*, η *Aquilæ*, and δ *Cephei*, for which satisfactory curves have already been obtained. The method of Argelander, or that proposed above, may be used with advantage.

It must be remembered that the observations will have little value, unless they are reduced and the light curve found. A vast number of excellent observations of these stars already exist, including the larger part of those of Argelander, which will have no value until they are reduced.

2. Determination of the light curve of the stars of the fourth class photometrically. This may be done with great accuracy by an instrument similar to that described in the Annals of the Harvard College Observatory, xi. 4, Figs. 1 and 2. The proximity of the companions render these objects especially suitable for photometric measurement.

3. Photometric measures of the comparison stars used in (1), of those used by previous observers, and the reduction of the observations by these measures to light intensities.

4. Search for variables of the fourth class, selecting from the Durchmusterung those fulfilling the conditions named above. They may be readily identified by their companions, and observed very rapidly by a

transit instrument, or small equatorial. The first of the three methods of estimating their light is to be recommended for this work. It is sufficiently precise, and the scale used each evening would be readily found from the Durchmusterung magnitudes of the great mass of the stars which would, probably, be invariable in light. Any interesting variable would be detected by observations on a few nights.

5. Measures of the position angles, distances, and magnitudes of the companions. The approximate places given from the Durchmusterung in Table XIII. could thus be corrected, and the blanks for southern stars filled. The magnitudes could best be measured by the photometer recommended in (2). Otherwise especial care should be taken that the light of the fainter star was not affected by the proximity of the brighter.

6. Observations of the color and spectrum of these stars, to decide which ones, if any, should be included in the second class.

7. Distribution of the light in the spectra of these stars, and also of those of the second class at their maxima and minima.

8. Computation by Jacobi's method of the true diameter of a liquid ellipsoid in equilibrium, having given the period of rotation and the ellipticity of the equator.

9. Computation of the Galactic latitude and longitude (or distance and direction from the pole of the Milky Way) of variables of Classes II. and IV., of the planetary and other gaseous nebulæ, and of stars whose spectrum is of the fourth type.

10. Computation of the position of the poles of the orbits of the binary stars.

The object of the present paper is not to advocate a certain theory which may seem improbable, and, possibly to some, inadequate. It is rather intended to bring together the most important facts bearing on the study of an interesting class of objects, and to exhibit them in a form in which they may be subjected to any desired test. The hypothesis advanced has a value as affording a simple geometrical conception of the nature of the variations under consideration, even if it proves not to be the true explanation of the cause. The ingenious hypothesis of Zöllner, and other explanations of these phenomena, have not been overlooked. It seemed best, however, to leave to another to decide the comparative merits of views in which the precision of the effects must be considered as well as the probability of the causes.

One theory, that the variation is due to the absorption of a rotating mass of gas, deserves a moment's consideration. This explanation does not appear probable for stars of the fourth class, since no evidence

of absorption is in general shown in their spectra beyond the appearance of lines such as are seen in our Sun. For the stars of the second class, however, this view seems more reasonable, since many of them exhibit spectra which are strongly banded. Moreover, the great variation in light is thus explained. An excellent test of this hypothesis is afforded by the variation in light of the different portions of the spectra. For light of any given wave-length the logarithm of the transmitted ray will always vary proportionally to the thickness and density of the absorbing medium, the amount of absorbent effect for any given thickness varying with the wave-length. Accordingly, a study of the variation of each ray should show the same law. They would give very different coefficients of absorptions, those of the dark bands being large, and those of the bright zones being small. The great variation in light will render this test a severe one with even a moderate degree of accuracy in the observations. For the lack of any data, this method of study is for the present unavoidably postponed.

The principal conclusions of the above paper may be summarized as follows: —

Thirty-one variable stars are known whose period is less than 72 days. Of those six belong to the fifth class, or that of β *Persei*, in which the variation is probably due to the interposition of an opaque eclipsing satellite. Of the remainder, seven may be excluded, since they are red, and may belong to the second class, or that of *o Ceti*. Nineteen remain, whose periods vary from less than a day to 54 days, and which may be placed in the fourth class. All lie within 16° of a great circle whose pole is in R. A. 13^h, Dec. + 20°. The distances of eleven are from 0° to 5°, of five at distances 8° and 9°, one at 14°, and one at 16°. The average distance is 5°.5, while if the stars were distributed at random it should be 30°.

If the stars of the Durchmusterung were uniformly distributed, their average distance apart would be about 8′.7. The five stars of the fifth class have Durchmusterung companions at an average distance of 10′.6. In the fourth class, excluding the red stars, six are in the Durchmusterung, and have companions at an average distance of 2′.5, four being less than 2′.0 distance, one at 3′.2, and one at 6′.1. In all six cases the direction of the companions is within less than 34° of the plane near which the variables lie, or at an average distance of 18°, while, if distributed by chance, this angle should be 45°. Hence a method of discovering variable stars of this class is offered by looking in a certain part of the sky for those having near companions in a given direction.

The light curves of four stars have been determined with sufficient precision to permit a comparison with theory. All of these may be represented by the formula $L' = 1 + m' \sin (v' + a) + n' \cos 2v'$, in which L' is the light when the star has turned through the angle v'. The difference between observation and theory amounts on the average to only about 0.03 of a magnitude. In other words, the light of these stars at any time may be computed with this degree of precision.

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